

Convergence of a Four-Step Iteration Process for G -nonexpansive Mappings in Banach Spaces with a Digraph

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Abstract: This review reckons with iterative scheme of Thianwan to approximate a common fixed point for four G -nonexpansive mappings (tersely G -nm). We verify several convergence results for in this way mappings in Banach space by dint of a digraph.

Keywords: Fixed point, digraph, G -nonexpansive mappings.

1. Introduction and preliminaries

Let X be a Banach space, $K \neq \emptyset \subseteq X$. *Directed graph* mostly enrolled qua *digraph* is a double: $G = (V(G), E(G))$, that here $V(G)$ is the set of vertices of graph and $E(G)$ is the set of its edges that involves overall the loops, scilicet $\forall x \in V(G); (x, x) \in E(G)$. Given that G enjoys no parallel edges. If x, y occur vertices of G , here a *path* in G ranging x from y of length N is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x = x_0, y = x_N$ and $(x_{i-1}, x_i) \in E(G)$ for $i = \overline{1, N}$. Digraph G is alleged to become *transitive* if, for $\forall x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, we acquire $(x, z) \in E(G)$ [2]. A mapping $f : K \rightarrow K$ is asserted to become

- G -nonexpansive (tersely G -nm) [3] if it yields (i) $(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$ (f preserves edges of G), (ii) $(x, y) \in E(G) \Rightarrow \|fx - fy\| \leq \|x - y\|$;
- *semi-compact* [9] if for $\{x_n\}$ in K with $\|x_n - fx_n\| \rightarrow 0$ as $n \rightarrow \infty$, there appears a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow f^* \in K$.

The mappings $f_i : K \rightarrow K$ are supply *condition* (A'') [1] if there is a nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0, 0 < g(t)$ for $\forall t \in (0, \infty)$ such that $\|x - f_i x\| \geq g(d(x, F_f))$ $i = \overline{1, k}; \forall x \in K$ where $d(x, F_f) = \inf \{\|x - f^*\| : f^* \in F_f = \cap_{c=1}^k F(f_c) \neq \emptyset\}$.

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Let $x_0 \in V(G)$ and $\Upsilon \subseteq V(G)$. We state that [5], (i) Υ is dominated by x_0 if $(x_0, x) \in E(G)$ for $\forall x \in \Upsilon$, (ii) Υ dominates x_0 if for each $x \in \Upsilon$, $(x_0, x) \in E(G)$.

Let G be a digraph such that $V(G) = K$. Then K is alleged to get *Property P* [8] if for each sequence $\{x_n\}$ in $K \rightarrow x \in K$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\forall l \in N; (x_{n_l}, x) \in E(G)$.

Remark 1.1 [6] *If G is transitive, then Property P is equal to the speciality: if $\{x_n\} \subseteq K$ with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_l}\}$ of $\{x_n\} \rightarrow x \subseteq X$, then $\forall n \in N; (x_n, x) \in E(G)$.*

Phuengrattana and Suantai [15] gave on the rate of convergence of Mann, Ishikawa, Noor and *SP*-iterations for continuous functions on an arbitrary interval. Sahin and Başarır [16] presented on the strong and Δ -convergence of *SP*-iteration on $CAT(0)$ space.

Motivated by [11], [12], [13] and above results, the iterative scheme is defined as follows:

$$\begin{aligned} t_n &= (1 - \beta_n)x_n + \beta_n f_1 x_n \\ y_n &= (1 - \xi_n)x_n + \xi_n f_2 t_n \\ s_n &= (1 - \varrho_n)y_n + \varrho_n f_3 y_n \\ x_{n+1} &= (1 - \theta_n)x_n + \theta_n f_4 s_n, \quad n \geq 1, \end{aligned} \tag{1}$$

where $\{\xi_n\}, \{\theta_n\}, \{\beta_n\}, \{\varrho_n\} \subseteq [0, 1]$, for $i = \overline{1, 4}$ $f_i : K \rightarrow K$ are G -nm. We verify several convergence results for in this way mappings in Banach space by dint of a digraph.

Lemma 1.2 [10] *Let X be a uniformly convex Banach space. Supposing that $0 < b \leq \nu_n \leq c < 1$, $n \geq 1$. Let $\{x_n\}, \{y_n\} \subseteq X$ be such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$ and $\lim_{n \rightarrow \infty} \|\nu_n x_n + (1 - \nu_n)y_n\| = a$, where $a \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

2. Main Results

$F_f = \cap_{c=1}^4 F(f_c) \neq \emptyset$. For $x_0 \in K$, let $\{x_n\}$ be the sequence created by (1).

Proposition 2.1 *Let $u_0 \in F_f$ be such that (x_0, u_0) and (u_0, x_0) are in $E(G)$. Then $\forall n \in N; (x_n, u_0), (u_0, x_n), (x_n, s_n), (s_n, x_n), (x_n, y_n), (y_n, x_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1})$ are in $E(G)$.*

Proof We shall demonstrate our deductions by induction. Let $(x_0, u_0) \in E(G)$. By virtue of edge-preserving of f_1 , we have $(f_1 x_0, u_0) \in E(G)$, and thus $(t_0, u_0) \in E(G)$ from the convexity of $E(G)$. Due to edge-preserving of f_2 , we get $(f_2 t_0, u_0) \in E(G)$. By using the convexity of $E(G)$

and $(x_0, u_0), (f_2 t_0, u_0) \in E(G)$, we own $(y_0, u_0) \in E(G)$. As f_3 is edge-preserving, we possess $(f_3 y_0, u_0) \in E(G)$ and $(s_0, u_0) \in E(G)$ from the convexity of $E(G)$. Owing to edge-preserving of f_4 , $(f_4 s_0, u_0) \in E(G)$. Again the convexity of $E(G)$ and $(x_0, u_0), (f_4 s_0, u_0) \in E(G)$, we acquire $(x_1, u_0) \in E(G)$. Continuing in this fashion for (x_1, u_0) instead of (x_0, u_0) , we get $(t_1, u_0), (y_1, u_0), (s_1, u_0), (x_2, u_0) \in E(G)$. Suppose that $(x_v, u_0) \in E(G)$ for $v \geq 1$. Because of edge-preserving of f_1 , we attain $(f_1 x_v, u_0) \in E(G)$, and thus $(t_v, u_0) \in E(G)$ from the convexity of $E(G)$. On account of edge-preserving of f_2 , we achieve $(f_2 t_v, u_0) \in E(G)$. Using the convexity of $E(G)$ and $(x_v, u_0), (f_2 t_v, u_0) \in E(G)$, we obtain $(y_v, u_0) \in E(G)$. Because f_3 is edge-preserving, we own $(f_3 y_v, u_0) \in E(G)$ and so $(s_v, u_0) \in E(G)$ from the convexity of $E(G)$. In view of edge-preserving of f_4 , $(f_4 s_v, u_0) \in E(G)$. Repetition the convexity of $E(G)$ and $(x_v, u_0), (f_4 s_v, u_0) \in E(G)$, we belong $(x_{v+1}, u_0) \in E(G)$. Repeating the procedure on one occasion for $(x_{v+1}, u_0) \in E(G)$, we get $(t_{v+1}, u_0), (y_{v+1}, u_0), (s_{v+1}, u_0), (x_{v+2}, u_0) \in E(G)$. Hence, $(x_n, u_0), (t_n, u_0), (y_n, u_0), (s_n, u_0) \in E(G)$ for $n \geq 1$. Utilizing an analog argumentum, we infer that $(u_0, x_n), (u_0, t_n), (u_0, y_n), (u_0, s_n) \in E(G)$ from $(u_0, x_0) \in E(G)$. As the graph G is transitivity, we acquire for $n \geq 1$ $(x_n, s_n), (s_n, x_n), (y_n, x_n), (x_n, y_n), (t_n, x_n), (x_n, t_n)$ and $(x_n, x_{n+1}) \in E(G)$. $\square$

Lemma 2.2 *If K is a nonempty closed convex subset of a real uniformly convex Banach space X , $\{\xi_n\}, \{\theta_n\}, \{\beta_n\}, \{\varrho_n\} \subseteq [a, b]$, where $0 < a < b < 1$ and $(x_0, u_0), (u_0, x_0) \in E(G)$ for $x_0 \in K$ and $u_0 \in F_f$, then*

(i) $\|x_{n+1} - u_0\| \leq \|x_n - u_0\|$ for $n \geq 1$, and hence $\|x_n - u_0\| \rightarrow 0$ as $n \rightarrow \infty$;

(ii) $\lim_{n \rightarrow \infty} \|x_n - f_i x_n\| = 0$ for $\overline{1, 4}$.

Proof (i) By Proposition 2.1, $(x_n, u_0), (u_0, x_n), (s_n, x_n), (x_n, s_n), (y_n, x_n), (x_n, y_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1})$ are in $E(G)$. It follows from (1) that

$$\begin{aligned}
 \|t_n - u_0\| &= \|-u_0 + (-\beta_n + 1)x_n + \beta_n f_1 x_n\| \\
 &\leq (-\beta_n + 1)\|-u_0 + x_n\| + \beta_n \|f_1 x_n - u_0\| \\
 &\leq (-\beta_n + 1)\|-u_0 + x_n\| + \beta_n \|-u_0 + x_n\| \\
 &= \|-u_0 + x_n\|.
 \end{aligned} \tag{2}$$

Using (1) & (2), we have

$$\begin{aligned}
\|y_n - u_0\| &\leq (1 - \xi_n) \|x_n - u_0\| + \xi_n \|f_2 t_n - u_0\| \\
&\leq (1 - \xi_n) \|x_n - u_0\| + \xi_n \|t_n - u_0\| \\
&\leq \|x_n - u_0\|.
\end{aligned} \tag{3}$$

Similarly, along with (3), we get

$$\begin{aligned}
\|s_n - u_0\| &\leq (1 - \varrho_n) \|y_n - u_0\| + \varrho_n \|f_3 y_n - u_0\| \\
&\leq (1 - \varrho_n) \|y_n - u_0\| + \varrho_n \|y_n - u_0\| \\
&\leq \|y_n - u_0\| \\
&\leq \|x_n - u_0\|.
\end{aligned} \tag{4}$$

By (4) we possess

$$\begin{aligned}
\|-u_0 + x_{n+1}\| &\leq (-\theta_n + 1) \|-u_0 + x_n\| + \theta_n \|-u_0 + f_4 s_n\| \\
&\leq (-\theta_n + 1) \|-u_0 + x_n\| + \theta_n \|s_n - u_0\| \\
&\leq \|x_n - u_0\|.
\end{aligned} \tag{5}$$

Hence, $\lim_{n \rightarrow \infty} \|x_n - u_0\|$ exists.

(ii) By assumption (i), $\{x_n\}$ is bounded. Let

$$\lim_{n \rightarrow \infty} \|x_n - u_0\| = M. \tag{6}$$

If $M = 0$, then, by $G - nm$ of $\{f_1, f_2, f_3, f_4\}$, it is obvious. Next, suppose $M > 0$. We shall show that for $i = \overline{1, 4}$ $\|x_n - f_i x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Getting $\limsup$ on both parts of (2), (3) & (4), we have

$$\limsup_{n \rightarrow \infty} \|t_n - u_0\| \leq M, \tag{7}$$

$$\limsup_{n \rightarrow \infty} \|y_n - u_0\| \leq M, \tag{8}$$

$$\limsup_{n \rightarrow \infty} \|s_n - u_0\| \leq M. \tag{9}$$

It implies by (7), (8), (9) and the $G - nm$ of $\{f_1, f_2, f_3, f_4\}$ that

$$\begin{aligned}
\|f_1 x_n - u_0\| &\leq \|x_n - u_0\| \\
\limsup_{n \rightarrow \infty} \|f_1 x_n - u_0\| &\leq M,
\end{aligned} \tag{10}$$

$$\begin{aligned} \|f_2 t_n - u_0\| &\leq \|t_n - u_0\| \\ \limsup_{n \rightarrow \infty} \|f_2 t_n - u_0\| &\leq M, \end{aligned} \quad (11)$$

$$\begin{aligned} \|f_3 y_n - u_0\| &\leq \|y_n - u_0\| \\ \limsup_{n \rightarrow \infty} \|f_3 y_n - u_0\| &\leq M, \end{aligned} \quad (12)$$

and

$$\begin{aligned} \|f_4 s_n - u_0\| &\leq \|s_n - u_0\| \\ \limsup_{n \rightarrow \infty} \|f_4 s_n - u_0\| &\leq M. \end{aligned} \quad (13)$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - u_0\| = M$, we get

$$\lim_{n \rightarrow \infty} \|(1 - \theta_n)(x_n - u_0) + \theta_n(f_4 s_n - u_0)\| = M. \quad (14)$$

By Lemma 1.2, we obtain

$$\|x_n - f_4 s_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (15)$$

Now, using the G - nm of $\{f_1, f_2, f_3, f_4\}$, we have

$$\begin{aligned} \|-u_0 + x_n\| &\leq \|f_4 s_n - u_0\| + \|-f_4 s_n + x_n\| \\ &\leq \|x_n - f_4 s_n\| + \|s_n - u_0\| \end{aligned} \quad (16)$$

$$\begin{aligned} &\leq \|x_n - f_4 s_n\| + \|(1 - \varrho_n)(y_n - u_0) + \varrho_n(f_3 y_n - u_0)\| \\ &\leq \|x_n - f_4 s_n\| + (1 - \varrho_n)\|y_n - u_0\| + \varrho_n\|f_3 y_n - u_0\| \\ &\leq \|x_n - f_4 s_n\| + \|y_n - u_0\| \end{aligned} \quad (17)$$

$$\begin{aligned} &\leq \|x_n - f_4 s_n\| + \|(1 - \xi_n)(x_n - u_0) + \xi_n(f_2 t_n - u_0)\| \\ &\leq \|x_n - f_4 s_n\| + (1 - \xi_n)\|x_n - u_0\| + \xi_n\|f_2 t_n - u_0\| \\ &\leq \frac{1}{\xi_n}\|x_n - f_4 s_n\| + \|t_n - u_0\| \\ &\leq \frac{1}{a}\|x_n - f_4 s_n\| + \|t_n - u_0\|. \end{aligned} \quad (18)$$

Taking $\liminf$ on both sides of (16), (17), (18) and using (15), we obtain

$$M \leq \liminf_{n \rightarrow \infty} \|s_n - u_0\|, \quad (19)$$

$$M \leq \liminf_{n \rightarrow \infty} \|y_n - u_0\|, \quad (20)$$

$$M \leq \liminf_{n \rightarrow \infty} \|t_n - u_0\|, \text{ resp.} \quad (21)$$

By combining (7) & (21), (8) & (20), (9) & (19), we get

$$\lim_{n \rightarrow \infty} \|t_n - u_0\| = \lim_{n \rightarrow \infty} \|y_n - u_0\| = \lim_{n \rightarrow \infty} \|s_n - u_0\| = M, \text{ resp.} \quad (22)$$

Namely,

$$\lim_{n \rightarrow \infty} \|(1 - \beta_n)(x_n - u_0) + \beta_n(f_1 x_n - u_0)\| = M,$$

$$\lim_{n \rightarrow \infty} \|(1 - \xi_n)(x_n - u_0) + \xi_n(f_2 t_n - u_0)\| = M,$$

$$\lim_{n \rightarrow \infty} \|(1 - \varrho_n)(y_n - u_0) + \varrho_n(f_3 y_n - u_0)\| = M, \text{ resp.}$$

It follows from (6), (8), (10), (11), (12) and Lemma 1.2 that

$$\lim_{n \rightarrow \infty} \|x_n - f_1 x_n\| = 0, \quad (23)$$

$$\lim_{n \rightarrow \infty} \|x_n - f_2 t_n\| = 0, \quad (24)$$

$$\lim_{n \rightarrow \infty} \|y_n - f_3 y_n\| = 0, \text{ resp.} \quad (25)$$

It implies by (23) & (24) that

$$\begin{aligned} \|x_n - f_2 x_n\| &\leq \|x_n - f_2 t_n\| + \|f_2 t_n - f_2 x_n\| \\ &\leq \|x_n - f_2 t_n\| + \|t_n - x_n\| \\ &\leq \|x_n - f_2 t_n\| + \beta_n \|f_1 x_n - x_n\| \\ &\leq \|x_n - f_2 t_n\| + b \|f_1 x_n - x_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (26)$$

By (1) & (24), we have

$$\begin{aligned} \|x_n - y_n\| &= \|x_n - [(1 - \xi_n)x_n + \xi_n f_2 t_n]\| \\ &\leq \xi_n \|x_n - f_2 t_n\| \\ &\leq b \|x_n - f_2 t_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (27)$$

It follows from (25) & (27), we get

$$\begin{aligned} \|x_n - f_3 x_n\| &\leq \|-y_n + x_n\| + \|y_n - f_3 y_n\| + \|f_3 y_n - f_3 x_n\| \\ &\leq \|-y_n + x_n\| + \|y_n - f_3 y_n\| \\ &\quad + \|-x_n + y_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (28)$$

By (1), (25) & (27), we have

$$\begin{aligned}
\|s_n - x_n\| &\leq \| -y_n + s_n \| + \|y_n - x_n\| \\
&= \|[(1 - \varrho_n)y_n + \varrho_n f_3 y_n] - y_n\| + \| -x_n + y_n \| \\
&\leq \varrho_n \|y_n - f_3 y_n\| + \| -x_n + y_n \| \\
&\leq b \|y_n - f_3 y_n\| + \| -x_n + y_n \| \\
&\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \tag{29}$$

Using (15) & (29), we obtain

$$\begin{aligned}
\|x_n - f_4 x_n\| &\leq \|x_n - f_4 s_n\| + \|f_4 s_n - f_4 x_n\| \\
&\leq \|x_n - f_4 s_n\| \\
&\quad + \|s_n - x_n\| \\
&\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \tag{30}$$

From (23), (26), (28) & (30), we get

$$\|x_n - f_i x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for } i = \overline{1, 4}. \tag{31}$$

□

Theorem 2.3 *Let K is a nonempty closed convex subset of a real uniformly convex Banach space X and $\{\xi_n\}, \{\theta_n\}, \{\beta_n\}, \{\varrho_n\} \subseteq [a, b]$, where $0 < a < b < 1$. Let $u_0 \in F_f$ such that $(x_0, u_0), (u_0, x_0)$ are in $E(G)$ for $x_0 \in K$. Supposing that K hold the Property P , $\{f_1, f_2, f_3, f_4\}$ satisfy the condition (A'') , F_f is dominated by x_0 and F_f dominates x_0 , then $\{x_n\} \rightarrow u_0 \in F_f$.*

Proof Let $u_0 \in F_f$ be such that for $\forall n \in N$ $(x_n, u_0), (u_0, x_n), (s_n, x_n), (x_n, s_n), (x_n, y_n), (y_n, x_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1})$ are in $E(G)$. Due to Lemma 2.2 (ii) and condition (A'') , we attain that $\lim_{n \rightarrow \infty} g(d(x_n, F_f)) = 0$. As g is nondecreasing with $g(0) = 0$, we hold $d(x_n, F_f) \rightarrow 0$ as $n \rightarrow \infty$. Thus, we can receive a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ and $\{u_l^*\} \subset F_f$ such that $\|x_{n_l} - u_l^*\| < 2^{-l}$. Due to the fact that strong convergence implies weak convergence and by Remark 1.1, we hold $(x_{n_l}, u_l^*) \in E(G)$. Using the proof method of [11], we own

$$\|x_{n_{l+1}} - u_l^*\| \leq \|x_{n_l} - u_l^*\| < \frac{1}{2^l},$$

and so

$$\| -u_{l+1}^* + u_l^* \| \leq \| -x_{n_{l+1}} + u_l^* \| + \| -u_{l+1}^* + x_{n_{l+1}} \| \leq 3 \cdot 2^{-(1+l)}.$$

We deduce that $\{u_{l+1}^*\}$ is a Cauchy sequence. Therefore, we have $u_l^* \rightarrow r$. By closed of F_f , $r \in F_f$ in that case $x_{n_l} \rightarrow r$. Because of Lemma 2.2 (i), $x_n \rightarrow r \in F_f$. $\square$

Theorem 2.4 *Let K is a nonempty closed convex subset of a real uniformly convex Banach space X and $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [a, b]$, where $0 < a < b < 1$. Let $u_0 \in F_f$ such that (x_0, u_0) , (u_0, x_0) are in $E(G)$ for $x_0 \in K$. Supposing that K has the Property P and one of $\{f_1, f_2, f_3, f_4\}$ is semi-compact, F_f is dominated by x_0 and F_f dominates x_0 , then $\{x_n\} \rightarrow u_0 \in F_f$.*

Proof Let $u_0 \in F_f$ be such that (x_n, u_0) , (u_0, x_n) , (x_n, s_n) , (s_n, x_n) , (x_n, y_n) , (y_n, x_n) , (x_n, t_n) , (t_n, x_n) , (u_0, s_n) , (s_n, u_0) , (u_0, y_n) , (y_n, u_0) , (u_0, t_n) , (t_n, u_0) , (x_n, x_{n+1}) are in $E(G)$ for $\forall n \in N$. We have $\lim_{n \rightarrow \infty} \|x_n - f_j x_n\| = 0$ from Lemma 2.2 (ii). Assume that f_j is semi-compact for $j = \overline{1, 4}$. Then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\lim_{l \rightarrow \infty} \|x_{n_l} - v\| = 0$ for some $v \in K$. This together with Remark 1.1 implies that $(x_{n_l}, v) \in E(G)$. It follows from the G -nm of $\{f_1, f_2, f_3, f_4\}$ and Lemma 2.2 (ii) that

$$\begin{aligned} \|v - f_j v\| &\leq \|v - x_{n_l}\| + \|x_{n_l} - f_j x_{n_l}\| + \|f_j x_{n_l} - f_j v\| \\ &\rightarrow 0 \text{ as } l \rightarrow \infty, \end{aligned}$$

for $j = \overline{1, 4}$. Hereat $v \in F_f$ so that $\lim_{n \rightarrow \infty} \|x_n - v\|$ exists. Thus, $x_n \rightarrow v$ as $n \rightarrow \infty$. $\square$

We indicate an instance which is inspired by Example 4.5 in [7].

Example 2.5 $K = [0, 2] \subseteq X = R$. Let G be a digraph described by $V(G) = K$ and $(x, y) \in E(G)$ iff $1.20 \geq y \geq x \geq 0.50$. Denote $\{f_1, f_2, f_3, f_4\} : K \rightarrow K$ by $f_1 x = 1 + \frac{23}{49} \tan(-1 + x)$, $f_2 x = 1 + \frac{29}{45} \tan(-1 + x)$, $f_3 x = 1 + \frac{23}{49} \arcsin(-1 + x)$, $f_4 x = 1 + \frac{29}{45} \arcsin(-1 + x)$ for any $x \in K$ and $i = 1, 2, 3, 4$. It is easy to see that f_1, f_2, f_3, f_4 are G -nm, but f_1, f_2, f_3, f_4 are not nonexpansive. Let $\beta_n = \frac{6n+5}{8n+15}$, $\xi_n = \frac{3n+1}{9n+20}$, $\varrho_n = \frac{10n+3}{11n+4}$, $\theta_n = \frac{7n+11}{13n+47}$ for $n \geq 1$. $F_f = \cap_{c=1}^4 F(f_c) = \{1\}$.

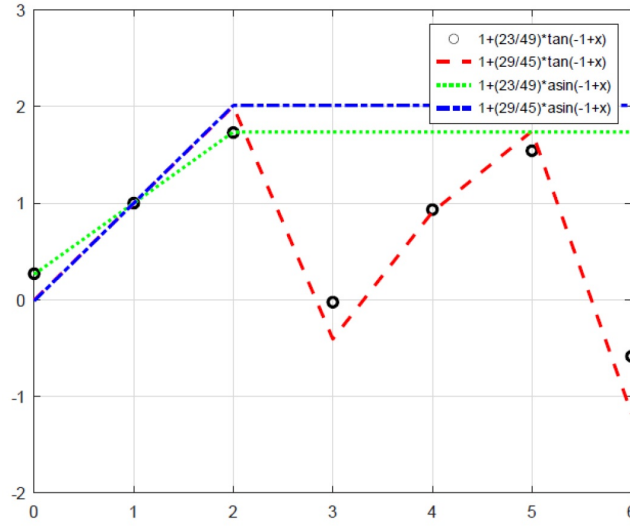
Figure 1: Plot showing $F_f = \cap_{c=1}^4 F(f_c) = \{1\}$

Table 1 The value of the sequence $\{x_n\}$ with initial value $x_0 = 1.20000$, $x_0 = 0.80000$ and $n = 20$, resp.

n	x_n	x_n
1	1.20000	0.80000
2	1.15950	0.84047
3	1.12180	0.87822
4	1.09010	0.90994
5	1.06500	0.93499
6	1.04600	0.95395
7	1.03210	0.96788
8	1.02210	0.97787
9	1.01510	0.98492
10	1.01020	0.98981
11	1.00680	0.99317
12	1.00450	0.99545
13	1.00300	0.99699
14	1.00200	0.99802
15	1.00130	0.99870
16	1.00090	0.99915
17	1.00060	0.99945
18	1.00040	0.99964
19	1.00030	0.99977
20	1.00020	0.99985

Remark 2.6 (i) If $\xi_n \equiv 0$ and $f_1 = f_2 = f_3 = f_4 = f$ in (1), then Theorem 2.3 generalize the results of Theorem 3.6 in [14] for self-map.

(ii) If $\xi_n = \varrho_n \equiv 0$ and $f_1 = f_2 = f_3 = f_4 = f$ in (1), we attain convergence of the Mann iteration to some fixed points of f on Banach space involving a digraph.

(iii) If $f_1 = f_2 = f_3 = f_4 = f$ in (1), then Theorem 2.3 extends the results of [12] without errors for self-map.

(iv) If $f_1 = f_2$, $f_3 = f_4$ in (1), then Theorem 2.3 improves the results of [13] without errors for self-map.

(v) If $\xi_n \equiv 0$ in (1), then Theorem 2.4 reduces to the results of [4].

Conclusion 2.7 In this writting, we reckons with four step iteration scheme to common fixed points of four G – nm described on Banach space involving a digraph. Our findings evolve the equal results of Shahzad (2005) [14], Thianwan (2008) [12], Kızıltunc et al. (2010) [13] and Tripak (2016) [4]. Within the future scope of the idea, reader can show that (1) compare convergence rate Picard, Mann, Ishikawa and SP –iteration process for contractions.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The authors declare no conflict of interest.

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