

SOME RESULTS ON G -CONTRACTION WITH PROPERTY (M)

Yolacan Esra

Abstract: Based on the results of Berinde and Păcurar, we present the notion of G_C - modified Reich contractions with property (M) in Banach spaces (curtly BS) including a graph. Our outcomes enlarge varied comparable consequences in the existing litterateur.

Keywords: Orbitally G -continuity, G -contraction, non-self map

MSC (2010): 47H09, 47H10, 54H25.

1. INTRODUCTION

In 2008, Jachymski [11] have given a catchy approach in theoric of fixed points in some overall forms by applying as part of metric spaces involving a graph. The author [11] has shown certain generalities on classical contraction mapping basis of Banach on a complete metric space through graph. Thereafter different contractions have been established by many researchers. In [12]-[10] and [15] the contraction basis for multivalued and set valued mappings, in [6], [7], [8] Reich and Kannan type contractions, Zamfirescu maps studied, resp. More precisely, let (X, d) be a metric space, $G = (V(G), E(G))$ be a graph such that $V(G) = X$, $E(G)$ subsumes entire loops. A map [11] $h : X \rightarrow X$ is orbitally G -continuity if $\forall r, s \in X, \{k_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Z}^+, h^{k_n} r \rightarrow s, (h^{k_n} r, h^{k_{n+1}} r) \in E(G) \Rightarrow h(h^{k_n} r) \rightarrow hs$.

Besides, Berinde and Păcurar [4] initiated study of fixed point theory for non-self contractions via Banach space (curtly BS) encorporing G . Balog and Berinde [2] established the notion of non-self Kannan contractions described on BS by of way G . Recently, a new type of non-self Catterjea is introduced and studied by Balog et al. [3]. Their results are more general than that of Berinde and Păcurar [4] and Balog and Berinde.

Inspired by former works, we define notion of modified non-self Reich type contractions on BS via G . The findings of this article enlarge varied known consequences in the literature.

2. ESSENTIAL DESCRIPTIONS AND FACTS IN BS VIA G

Let X be BS, $C \neq \emptyset$ a closed of X , $h : C \rightarrow X$ a non-self map. If $r \in C$ is such that $hr \notin C$, here we mostly select $s \in \partial C$ such that $s = (1 - \mu)r + \mu hr (\mu \in (0, 1))$, which fairly express

$$d(hr, r) = d(hr, s) + d(s, r), \quad s \in \partial C. \quad (2.1)$$

The set B of points s supplying (2.1) can compass more than one element. Assume that $B \neq \emptyset$.

Definition 2.1. [5] Let X be BS, $C \neq \emptyset$ a closed of X , $h : C \rightarrow X$ a non-self map. Let $r \in C$ with $hr \notin C$, $s \in \partial C$ be the concerned elements endowed with (2.1). If, for any such terms r , we obtain

$$d(s, hs) \leq d(r, hr) \quad (2.2)$$

for all corresponding $s \in B$, then h enjoy *property (M)*.

Definition 2.2. [4] Let X be BS, $C \neq \emptyset$ a closed of X . A map $h : C \rightarrow X$ is (i) well defined, if it holds the quality for the subgraph of G induced by C , viz, $\forall r, s \in C$,

$$(r, s) \in E(G) \text{ with } hr, hs \in C \Rightarrow (hr, hs) \in E(G) \cap (C \times C). \quad (2.3)$$

(ii) G_C -contraction if h is (well) defined and there exists $z_1 \in [0, 1)$ such that $\forall r, s \in C$,

$$(r, s) \in E(G_C) \Rightarrow d(hr, hs) \leq z_1 d(r, s). \quad (2.4)$$

Definition 2.3. (*Property (L)*) [4] Any sequence for $\forall n \in \mathbb{N}, \{r_n\} \subseteq X$ with $r_n \rightarrow r$ when $n \rightarrow \infty$ and $(r_n, r_{n+1}) \in E(G)$, there appears a subsequence $\{r_{k_n}\}_{i=1}^{\infty}$ providing $(r_{k_n}, r) \in E(G)$.

3. MAIN RESULTS

Theorem 3.1. *Let (X, d, G) be BS via weakly connected and simple directed G such that the property (L) acquires. Let $C \neq \emptyset$ be a closed of X , $h : C \rightarrow X$ be G_C -modified Reich contraction with property (M), namely, there appears η, ϑ, θ , nonnegative real numbers with $\eta + \vartheta + \theta < 1$ such that for $\forall (r, s) \in E(G_C)$*

$$d(hr, hs) \leq \eta d(r, s) + \vartheta d(r, hr) + \theta d(s, hs) + U_R(r, s), \quad (3.1)$$

where

$$U_R(r, s) = R \min \{d(r, s), d(r, hr), d(s, hs), d(r, hs), d(s, hr)\}, R \geq 0$$

and G_C is the subgraph of G stated by C . If $C_h := \{r \in \partial C : (r, hr) \in E(G)\} \neq \emptyset$, h getting Rothe's boundary condition

$$C \supset h(\partial C), \quad (3.2)$$

then h own a fixed point in C .

Proof. Let $C \supseteq h(C)$. Here h is substantially a self-map in closed set C , the consequence pursued by Reich fixed point theorem [13] to $X = C$. Hence, we only take into consideration $h(C) \not\subseteq C$. Moreover, we have $\forall n \in \mathbb{N}$

$$r_0 \in C_h \Rightarrow (r_0, hr_0) \in E(G) \Rightarrow (h^n r_0, h^{n+1} r_0) \in E(G). \quad (3.3)$$

Express $\forall n \in \mathbb{N}; s_n := h^n r_0$.

Owing to (3.2), we get that $hr_0 \in C$.

Denote $r_1 := s_1 = hr_0$. Next, if $hr_1 \in C$, set $r_2 := s_2 = hr_1$. If $hr_1 \notin C$, we may seize a term r_2 on the segment $[r_1, hr_1]$ which also pertain to ∂C , i.e.,

$$r_2 = (1 - \mu)r_1 + \mu hr_1 (\mu \in (0, 1)).$$

Ongoing argumentums we create two sequences $\{r_n\}$ and $\{s_n\}$ whose elements fulfill one of the below particulars:

- (1) $hr_{n-1} \in C \Rightarrow r_n := s_n = hr_{n-1}$;
- (2) $hr_{n-1} \notin C \Rightarrow r_n = r_{n-1}(-\mu + 1) + \mu hr_{n-1} \in \partial C (\mu \in (0, 1))$.

Next, we allege that $\{r_n\}$ is a Cauchy sequence. Assume that

$$\begin{aligned} P &= \{r_k \in \{r_n\} : r_k = s_k = hr_{k-1}\}, \\ Q &= \{r_k \in \{r_n\} : r_k \neq hr_{k-1}\}. \end{aligned}$$

Clearly, if $r_k \in Q$, then r_{k-1} and r_{k+1} pertain to P . Furthermore, due to (3.2), we can not hold two consecutive nomials of $\{r_n\} \subseteq Q$ (yet we can hold two consecutive nomials of $\{r_n\} \subseteq P$). Next, we attach 3 possibilities.

Case 1. $r_n, r_{n+1} \in P$. Then, we get $s_n = hr_{n-1} = r_n, s_{n+1} = hr_n = r_{n+1}$. Thus

$$d(r_n, r_{n+1}) = d(hr_{n-1}, hr_n) = d(s_n, s_{n+1}).$$

As $\{r_n\} \subset C, \forall n \in \mathbb{N}$, from (3.3) $(r_n, r_{n-1}) \in E(G_C)$, and thus by G_C -modified Reich contraction condition (3.1), we obtain that

$$\begin{aligned} d(r_n, r_{n+1}) &= d(hr_{n-1}, hr_n) \\ &\leq \eta d(r_n, r_{n-1}) + \vartheta d(r_n, hr_n) + \theta d(r_{n-1}, hr_{n-1}) \\ &\quad + R \min \{d(r_{n-1}, r_n), d(hr_n, r_n), d(hr_{n-1}, r_{n-1}), d(hr_{n-1}, r_n), d(hr_n, r_{n-1})\} \\ &\leq \eta d(r_n, r_{n-1}) + \vartheta d(r_n, r_{n+1}) + \theta d(r_{n-1}, r_n) \\ &\quad + R \min \left\{ \begin{array}{l} d(r_n, r_{n-1}), d(r_n, r_{n+1}), \\ d(r_{n-1}, r_n), d(r_n, r_n), d(r_{n+1}, r_{n-1}) \end{array} \right\} \\ &\leq \eta d(r_n, r_{n-1}) + \vartheta d(r_n, r_{n+1}) + \theta d(r_{n-1}, r_n), \end{aligned}$$

$$d(r_n, r_{n+1}) \leq \kappa d(r_{n-1}, r_n) \quad (3.4)$$

where $\kappa = \frac{\eta+\vartheta}{1-\vartheta} < 1$.

Case 2. $r_{n+1} \in Q, r_n \in P$. Then, we have $s_n = r_n = hr_{n-1}, r_{n+1} \neq s_{n+1} = hr_n$. We get

$$\begin{aligned} d(r_{n+1}, r_n) &\leq d(r_{n+1}, r_n) + d(s_{n+1}, r_{n+1}) \\ &= d(s_{n+1}, r_n) \\ &= d(s_{n+1}, s_n). \end{aligned}$$

Next, using an analogue argumentum to Case 1, $(r_n, r_{n-1}) \in E(G_C)$ and so by (3.1), we attain

$$\begin{aligned} d(s_n, s_{n+1}) &= d(hr_{n-1}, hr_n) \\ &\leq \eta d(r_{n-1}, r_n) + \vartheta d(r_{n-1}, hr_{n-1}) + \theta d(r_n, hr_n) \\ &\quad + R \min \{d(r_{n-1}, r_n), d(r_{n-1}, hr_{n-1}), d(r_n, hr_n), d(r_{n-1}, hr_n), d(r_n, hr_{n-1})\} \\ &= \eta d(r_{n-1}, r_n) + \vartheta d(r_{n-1}, hr_{n-1}) + \theta d(s_n, s_{n+1}) \\ &\quad + R \min \left\{ \begin{array}{l} d(r_{n-1}, r_n), d(r_{n-1}, r_n), \\ d(s_n, s_{n+1}), d(r_{n-1}, r_{n+1}), d(r_n, r_n) \end{array} \right\} \\ &= \eta d(r_{n-1}, r_n) + \vartheta d(r_{n-1}, hr_{n-1}) + \theta d(s_n, s_{n+1}) \end{aligned}$$

which yields

$$\xi d(r_{n-1}, r_n) \geq d(s_n, s_{n+1}).$$

where $\xi = \frac{\eta+\vartheta}{1-\theta} < 1$ and so,

$$d(r_{n+1}, r_n) < d(s_{n+1}, s_n) \leq \xi d(r_{n-1}, r_n). \quad (3.5)$$

Case 3. $r_{n+1} \in P, r_n \in Q$. Then, we have $r_{n-1} \in P, s_n \neq r_n, s_{n+1} = r_{n+1}, s_{n-1} = r_{n-1}, s_n = hr_{n-1}$ and

$$d(r_{n-1}, r_n) + d(r_n, hr_{n-1}) = d(r_{n-1}, hr_{n-1}).$$

From the *property (M)*, we have

$$d(r_n, r_{n+1}) = d(r_n, hr_n) \leq d(r_{n-1}, hr_{n-1}) = d(hr_{n-2}, hr_{n-1}).$$

Therefore,

$$d(r_n, r_{n+1}) \leq d(hr_{n-2}, hr_{n-1}).$$

As $\{s_n\} \subseteq C, \forall n \in \mathbb{N}$, from (3.3) $(r_{n-2}, r_{n-1}) \in E(G_C)$, and thus by (3.1), we obtain that

$$\begin{aligned} d(hr_{n-2}, hr_{n-1}) &\leq \eta d(r_{n-2}, r_{n-1}) + \vartheta d(r_{n-2}, hr_{n-2}) + \theta d(r_{n-1}, hr_{n-1}) \\ &\quad + R \min \left\{ \begin{array}{l} d(r_{n-2}, r_{n-1}), d(r_{n-2}, hr_{n-2}), \\ d(r_{n-1}, hr_{n-1}), d(r_{n-2}, hr_{n-1}), d(r_{n-1}, hr_{n-2}) \end{array} \right\} \\ &= \eta d(r_{n-2}, r_{n-1}) + \vartheta d(r_{n-2}, r_{n-1}) + \theta d(r_{n-1}, r_n) \\ &\quad + R \min \left\{ \begin{array}{l} d(r_{n-2}, r_{n-1}), d(r_{n-2}, r_{n-1}), \\ d(r_{n-1}, r_n), d(r_{n-2}, r_n), d(r_{n-1}, r_{n-1}) \end{array} \right\}, \end{aligned}$$

and thus,

$$d(hr_{n-2}, hr_{n-1}) = d(r_{n-1}, hr_{n-1}) \leq (\eta + \vartheta) d(r_{n-2}, r_{n-1}) + \theta d(r_{n-1}, hr_{n-1}),$$

which yields

$$d(r_{n-1}, hr_{n-1}) \leq \frac{\eta + \vartheta}{1 - \theta} d(r_{n-2}, r_{n-1}), \quad n \geq 2.$$

Hence,

$$\xi d(r_{n-2}, r_{n-1}) \geq d(r_n, r_{n+1}). \quad (3.6)$$

Now, by combining (3.4), (3.5) and (3.6), it follows that

$$d(r_n, r_{n+1}) \leq \begin{cases} \varrho d(r_n, r_{n-1}) \\ \varrho d(r_{n-2}, r_{n-1}) \end{cases}$$

where

$$\varrho = \max \left\{ \frac{\eta + \theta}{1 - \vartheta}, \frac{\eta + \vartheta}{1 - \theta} \right\} = \max \{ \kappa, \xi \}.$$

Following [4] from inductive, it concludes that for $n \geq 2$

$$d(r_n, r_{n+1}) \leq \varrho^{[n/2]} \sigma, \quad (3.7)$$

where

$$\sigma = \max \{ d(r_1, r_0), d(r_1, r_2) \}. \quad (3.8)$$

Here, $[n/2]$ stand for the largest \mathbb{Z} not surpassing $n/2$.

Next, for $N < n < m$;

$$d(r_n, r_m) \leq \sum_{j=N}^{\infty} d(r_j, r_{j-1}) \leq 2 \frac{\varrho^{[n/2]}}{1 - \varrho} \sigma.$$

This implies that $\{r_n\}$ is Cauchy sequence.

As $\{r_n\} \subset C$, $\forall n \in \mathbb{N}$ and C is closed, $\{r_n\}$ converges to $r_* \in C$, that is,

$$r_n \rightarrow r_* \text{ as } n \rightarrow \infty.$$

From the *property (L)* and (3.1), we get

$$d(hr_{k_n}, hr_*) \leq \eta d(r_{k_n}, r_*) + \vartheta d(r_{k_n}, hr_{k_n}) + \theta d(r_*, hr_*). \quad (3.9)$$

Hence, owing to (3.7) and (3.9)

$$\begin{aligned} d(r_*, hr_*) &\leq d(r_*, r_{k_n+1}) + d(r_{k_n+1}, hr_*) \\ &= d(r_*, r_{k_n+1}) + d(hr_{k_n}, hr_*) \\ &= d(r_*, r_{k_n+1}) + \eta d(r_{k_n}, r_*) + \vartheta d(r_{k_n}, hr_{k_n}) + \theta d(r_*, hr_*) \\ &\quad + R \min \left\{ \begin{array}{l} d(r_{k_n}, r_*), d(r_{k_n}, hr_{k_n}), \\ d(r_*, hr_*), d(r_{k_n}, hr_*), d(r_*, hr_{k_n}) \end{array} \right\} \\ &= d(r_*, r_{k_n+1}) + \eta d(r_{k_n}, r_*) + \vartheta d(r_{k_n}, hr_{k_n}) + \theta d(r_*, hr_*), \\ \\ d(r_*, hr_*) &\leq \frac{1}{1 - \theta} d(r_*, r_{k_n+1}) + \frac{\eta}{1 - \theta} d(r_{k_n}, r_*) + \frac{\vartheta}{1 - \theta} d(r_{k_n}, hr_{k_n}) \\ &\leq \frac{1}{1 - \theta} d(r_*, r_{k_n+1}) + \nu d(r_{k_n}, r_*) + \varrho d(r_{k_n}, r_{k_n+1}) \\ &\leq \frac{1}{1 - \theta} d(r_*, r_{k_n+1}) \\ &\quad + \nu \left[d(r_{k_n}, r_*) + \varrho^{[k_n/2]} \max \{ d(r_{k_0}, r_{k_1}), d(r_{k_1}, r_{k_2}) \} \right]. \end{aligned} \quad (3.10)$$

Taking the limit as $n \rightarrow \infty$, then $d(r_*, hr_*) = 0$ and hence $r_* = fr_*$.

To display that r_* is a unique fixed point of h , given that r_* , s_* are two fixed point of h . In that case, we get

$$\begin{aligned} d(r_*, s_*) &= d(hr_*, hs_*) \\ &\leq \eta d(r_*, s_*) + \vartheta d(r_*, hr_*) + \theta d(s_*, hs_*) \\ &\quad + R \min \{d(r_*, s_*), d(r_*, hr_*), d(s_*hs_*), d(r_*, hs_*), d(s_*, hr_*)\} \\ &= \eta d(r_*, s_*), \end{aligned}$$

and so

$$d(r_*, s_*) = 0 \text{ in view of } (1 - \eta) < 1.$$

It follows that $r_* = s_*$. □

Example 3.2. Let $X = [0, 1] \cup \{2, 3\}$ be endowed with the usual norm and let $C = \{0, 1, 2, 3\}$. Define the operator $h : C \rightarrow X$ hereinbelow

$$hr = \begin{cases} 0, & \text{if } r \in \{0, 1\}, \\ \frac{1}{5}, & \text{if } r \in \{2, 3\}. \end{cases}$$

Let G be defined by

$$E(G) = \{(r, r) : r \in [0, 1]\} \cup \left\{ (0, 1), (0, 2), (2, 3), (2, 2), \left(2, \frac{1}{5}\right), (3, 3) \right\}.$$

Then G is weakly connected graph such that the *property (L)* acquires and h is G_C -modified Reich contraction on C . It is also simple to overhaul that h is well defined on X via G_C . Actually, G_C own the set of vertices $E(G_C) = \{(3, 3), (0, 0), (2, 3), (0, 1), (2, 2), (0, 2), (1, 1)\}$. Due to (2.3), the edges $(0, 2), (2, 2), (2, 3), (3, 3)$ have to be endured and for the remaining edges we obtain

$$(h0, h0) = (h0, h1) = (h1, h1) = (0, 0) \in E(G_C).$$

Furthermore $C \supset h(\partial C)$ is provided, due to $\partial C = \{0, 1\}$, thus $h(\partial C) = \{0\} \subset C$. Eventually, because we again hold $C_h = \{0\} \neq \emptyset$, whole assumption in Theorem 3.1 are ensured, 0 is single fixed point of h .

Remark 3.3. (i) For $R = 0$ in Theorem 3.1, then it is easily proved G_C -Reich contractions with property (M) on BS including a graph.

(ii) Taking $\eta = 0$ and $\vartheta = \theta$ in Theorem 3.1 we obtain Kannan type contractions by Balog and Berinde Theorem 2.1 in [2] for a non-self map.

(iii) If $\vartheta = 0$ and $\theta = 0$ in Theorem 3.1, then it can be simply showed Theorem 3.1 in [4].

Like in [4, 2], we can replace the property (L) of (X, d, G) by favorable terms.

Theorem 3.4. Let (X, d, G) be BS via a weakly connected and simple directed G . Let $C \neq \emptyset$ be a closed of X , $h : C \rightarrow X$ be G_C -modified Reich contraction on C . If $C_h := \{r \in \partial C : (r, hr) \in E(G)\} \neq \emptyset$, h is orbitally G -continuous and h satisfies $C \supset h(\partial C)$, then h has a fixed point in C .

Proof. With respect to argument of Theorem 3.1, we have that $\{r_n\}$ defined by $r_n = h^n r_0$ for all n converges to r_* . Since $(h^n r_0, h^{n+1} r_0) \in E(G)$, $\forall n \in \mathbb{N}$, h is orbitally G -continuous, we obtain

$$r_* = \lim_{n \rightarrow \infty} h(h^n r_0) = hr_*.$$

That is, $r_* = hr_*$. Assume that s_* is another fixed point of h . Using an analogue technique as in Theorem 3.1, we get $r_* = s_*$. □

Remark 3.5. (i) For $R = 0$ in Theorem 3.4, then it is easily proved G_C -Reich contractions on BS including a graph.

(ii) Taking $\eta = 0$ and $\vartheta = \theta$ in Theorem 3.4, we obtain Kannan type contractions by Balog and Berinde Theorem 2.2 in [2] for a non-self map.

(iii) If $\vartheta = 0$ and $\theta = 0$ in Theorem 3.4, then it can be simply showed Theorem 3.2 in [4].

In the instant writing, non-self G_C -modified Reich contractions on BS via G have been defined. Our deductions conduce a more general approximation to such a non-self contractions engendered by [4],[2].

REFERENCES

- [1] Agwu I.K.; Convergence theorems of two multivalued mappings satisfying the jointly demiclosedness principle in Banach spaces. *Canad. J.Appl.Math.*,-2021.-3.-No1.-P.61-79.
- [2] Balog L., Berinde V., Fixed point theorems for nonself Kannan type contractions in Banach spaces endowed with a graph. *Carpathian J. Math.*,-2016.-32.-P. 293-302.
- [3] Balog L., Berinde V., Păcurar M., Approximating fixed points of nonself contractive type mappings in Banach spaces endowed with graph. *AN STI U OVID CO-MAT.*, -2016.-24.-No2. -P.27-43.
- [4] Berinde V., Păcurar M., The contraction principle for nonself mappings on Banach spaces endowed with a graph. *J Nonlinear Convex Anal.*,-2015 .-16.-No9. -P.1925-1936.
- [5] Berinde V., Păcurar M., Fixed point theorems for nonself single valued almost contractions. *Fixed Point Theory.*,-2013.-14.-No2.-P. 301-312.
- [6] Bojor F.; Fixed point theorems for Reich type contractions on metric space with a graph. *Nonlinear Anal Theory Methods Appl.*,-2012.-75.-No9.-P.3895-3901.
- [7] Bojor F., Fixed points of Kannan mappings in metric spaces endowed with a graph. *AN STI U OVID CO-MAT.*, -2012.-20.-No1. -P.31-40.
- [8] Bojor F., Tilca M., Fixed point theorems for Zamfirescu mappings in metric spaces endowed with a graph. *Carpathian J. Math.*, -2015.-31. -No3. -P.297-305.
- [9] Debnath P., Choudhury B.S., Neog M.; Fixed set of set valued mappings with set valued domain in terms of start set on a metric space with a graph. *Fixed Point Theory Appl.*,-2017.-5.
- [10] Ghods M., Ghobadi S., Fixed point results for single and multivalued mappings with application in graph theory. *J. Interdiscip. Math.*, -2022.-25.-No8-B.-P.2445-2455.
- [11] Jachymski J.; The contraction principle for mappings on a metric space with a graph. *Proc Am Math Soc* .,-2008.-136.-P.1359-1373.
- [12] Neog M., Debnath P., Radenovic S.; Common fixed point of set valued graph A_φ -contraction pair and generalized φ -weak G -contraction on metric space endowed with a graph. *Ann. Univ. Craiova Math. Comput. Sci. Ser.*,-2020.-47.-No1.-P.158-169.
- [13] Reich S.; Some remarks concerning contraction mappings. *Can. Math. Bull.*,-1971.-14.-P.121-124.
- [14] Shagari M.S., Alotaibi T., Aydi H., Park C.; Fixed point of nonlinear multivalued graphic contractions with applications. *AIMS Math.*,-2022.-7.-No11.-P.20164-20177.
- [15] Yolacan E., A Brief Note concerning non-self contractions in Banach Spaces endowed with a graph. *Gen. Lett. Math.*, -2017.-3.-No1. -P.25-30.

Yolacan E. ,
 Cappadocia University, School of Applied Sciences,
 Department of Airframe And Powerplant Maintenance,
 Mustafapaşa Campus
 50420 - Mustafapaşa, Ürgüp / Nevşehir-Türkiye
<https://orcid.org/0000-0002-1655-0993>
 email: yolacanesra@gmail.com