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Some Convergence Results of Iteratıon Scheme For $G-$ Asymptotically Nonexpansiveness

Esra YOLAÇAN[1]

Introduction and Preliminaries

Let $\eta \neq \emptyset$ be a subset of a Banach space X. A digraph is a pair: $G = (V(G), E(G))$, here $V(G)$ is the set of vertices of graph and $E(G)$ is the set of its edges which encapsules all the loops, i.e. $(p,p) \in E(G)$ for $\forall p \in V(G)$. Supposing G has no parallel edges. If p and q be vertices of G, then a path on G from p through q of length N is $\{p_i\}_{i=0}^{N}$ of $N+1$ vertices such that $p_0 = p$, $p_N = p$ and $(p_{i-1}, p_i) \in E(G)$ for $i = 1, \cdots, N$. A digraph G is called to be transitive if, for any $p, h, l \in V(G)$ such that $(p,h), (h,l) \in E(G)$, we have $(p,l) \in E(G)$.

The map $f: \eta \to \eta$ is called to be

- $G-$*nonexpansive* if it provides (i) $(p,l) \in E(G) \Rightarrow (fp, fl) \in E(G)$ (f preserves edges of G), (ii) $(p,l) \in E(G) \Rightarrow \|fp - fl\| \leq \|p - l\|$ (Alfuraidan&Khamsi, 2015);
- $G-$continuous if for any given $\tau \in X, \{\tau_n\} \subseteq X$, $\tau_n \longrightarrow \tau$ and $(\tau_n, \tau_{n+1}) \in E(G)$ imply $f\tau_n \to f\tau$ (Jachymski, 2008);
- *semicompact* if for $\{x_n\} \subseteq \eta$ with $\|x_n - fx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \sigma_* \in \eta$ (Shahzad&Al-Dubiban, 2006);
- $G-$ *asymptotically nonexpansive* if it provides (i) f preserves edges of G, (ii) there exists a sequence $\phi_n \subset [1, \infty), \phi_n \to 1$ as $n \to \infty$ such that $\|f^n p - f^n l\| \leq \phi_n \|p - l\|$, whenever $(p,l) \in E(G)$ for any $p, l \in \eta$ and $n \geq 1$ (Sangago & et al., 2018).

Fixed point (brief, *FP*) iterative process for $G-$nonexpansiveness maps in Banach spaces with a graph (brief, *BSWG*) including Ishikawa, $S-$iteration, explicit iteration, $SP-$iteration schemes have been intensively investigated by various researchers (Tripak, 2016), (Suparatulatorn& et al., 2018), (Hunde& et al., 2017), (Sridarat& et al., 2018). The authors established the class of $G-$asymptotically nonexpansiveness maps, who verified some convergence results in *BSWG* (Sangago & et al., 2018).

The mapping $f: \eta \to \eta$ is called to yield *Condition (A)* if there is a nondecreasing function $g: [0, \infty) \to [0, \infty)$ with $g(t) > 0$ for $\forall t \in (0, \infty)$, $g(0) = 0$ such that $\|x - fx\| \geq g\left(d\left(x, F(f)\right)\right)$. Two mappings $f_1, f_2: \eta \to \eta$ are called to yield *Condition (B)* if there is a nondecreasing function $g: [0, \infty) \to [0, \infty)$ with $g(t) > 0$ for $\forall t \in (0, \infty)$, $g(0) = 0$ such that $max\{\|x - f_1 x\|, \|x - f_2 x\|\} \geq g\left(d\left(x, Fix(F)\right)\right)$, where $Fix(F) = F(f_1) \cap F(f_2)$. The mappings $f_1, f_2, f_3: \eta \to \eta$ are called to yield *Condition (C)* if there is a nondecreasing function $g: [0, \infty) \to [0, \infty)$ with $g(t) > 0$ for $\forall t \in (0, \infty)$, $g(0) = 0$ such that $max\{\|x - f_1 x\|, \|x - f_2 x\|, \|x - f_3 x\|\} \geq g\left(d\left(x, Fix(F)\right)\right)$, where $Fix(F) = F(f_1) \cap F(f_2) \cap F(f_3)$.

Let $p_0 \in V(G)$ and $V(G) \supseteq \Gamma$. We call that (i) Γ is dominated by p_0 if $(p_0, p) \in E(G)$ for $\forall p \in \Gamma$, (ii) Γ dominates p_0 if for each $p \in \Gamma$, $(p_0, p) \in E(G)$ (Suparatulatorn& et al., 2018).

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Let $\eta \neq \emptyset \subseteq X$, $f\colon \eta \longrightarrow X$ be a map. Then is called to be $G-demiclosed$ at $q^* \in X$ [14] if, for any $\{x_n\} \subseteq \eta$ such that $\{x_n\} \rightharpoonup p^* \in \eta$, $\{fx_n\} \to q^*$ and $(x_n, x_{n+1}) \in E(G)$ imply $fp^* = q^*$.

Let $\eta \neq \emptyset \subseteq X$, $G = (V(G), E(G))$ be digraph such that $V(G) = \eta$. Here η is called to own *Property P* if for each $\{p_n\} \subseteq \eta$ such that $\{p_n\} \rightharpoonup p \in \eta$, $(p_n, p_{n+1}) \in E(G)$, there is a subsequence $\{p_{n_l}\}$ of $\{p_n\}$ such that $(p_{n_l}, p) \in E(G)$ for $\forall l \in N$ (Alfuraidan, 2015).

Remark 1. If G is transitive, then *Property P* is equivalence to the feature:

- If $\{p_n\} \subseteq \eta$ with $(p_n, p_{n+1}) \in E(G)$ such that for any subsequence $\{p_{n_l}\}$ of $\{p_n\}$ converging weakly to p in X, then $(p_n, p) \in E(G)$ for $\forall n \in N$ (Sridarat& et al., 2018).

The authors considered the following modified Ishikawa process, for arbitrary $x_0 \in \eta$,

$$x_{n+1} = (1-a_n)fx_n + a_n f\big((1-b_n)x_n + b_n fx_n\big), n \geq 1,$$

where $\{a_n\}$, $\{b_n\} \subset (0,1)$ (Agarwal & et al., 2007).

The authors introduced the following iteration scheme, for arbitrary $x_0 \in \eta$,

$$\begin{gathered} x_{n+1} = (1-a_n)fy_n + a_n fz_n, \\ y_n = (1-b_n)fx_n + b_n fz_n, \\ z_n = (1-c_n)x_n + c_n fx_n \; n \geq 1, \end{gathered}$$

where $\{a_n\}$, $\{b_n\}, \{c_n\} \subset (0,1)$ (Abbas&Nazir, 2014).

The authors studied an iterative process, for arbitrary $x_0 \in \eta$,

$$\begin{gathered} x_{n+1} = (1-a_n)fx_n + a_n fy_n, \\ y_n = (1-b_n)z_n + b_n fz_n, \\ z_n = (1-c_n)x_n + c_n fx_n \;, n \geq 1, \end{gathered}$$

where $\{a_n\}$, $\{b_n\}, \{c_n\} \subset (0,1)$ (Thakur& et al, 2014).

Motivated by above studies, in this writing we consider an iterative procedure for providing a common fixed point of three $G-$ asymptotically nonexpansive maps as follows:

For arbitrary $x_0 \in \eta$, $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1} &= (1-a_n)f_2^n x_n + a_n f_3^n y_n \\ y_n &= (1-b_n)z_n + b_n f_2^n z_n, \\ z_n &= (1-c_n)x_n + c_n f_1^n x_n \; n \geq 1, \end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}, \{c_n\} \subset (0,1)$.

Lemma 1. Let $\{\tau_n\}$, $\{\upsilon_n\}, \{\varrho_n\}$ be sequences of nonnegative real numbers supplying the inequality

$$\tau_{n+1} \leq (1+\varrho_n)\tau_n + \upsilon_n, n \geq 1$$

if $\sum_{n=1}^{\infty} \upsilon_n < \infty$ and $\sum_{n=1}^{\infty} \varrho_n < \infty$, then

(i) $\lim_{n\to\infty} \tau_n$ exists;

(ii) Notedly, if $\{\tau_n\}$ hold a subsequence $\{\tau_{n_k}\} \to 0$, then $\lim_{n\to\infty} \tau_n = 0$ (Tan&Xu,1993).

Lemma 2. Let X be a uniformly convex Banach space. Supposing $n \geq 1$, $1 > c \geq t_n \geq b > 0$. Let $\{e_n\}, \{h_n\} \subseteq X$ be such that $limsup_{n\to\infty}\|e_n\| \leq a$, $limsup_{n\to\infty}\|h_n\| \leq a$, $\|(1-t_n)h_n + t_n e_n\| \to a \geq 0$ as $n \to \infty$. Then $\|e_n - h_n\| \to 0$ as $n \to \infty$ (Sahu, 1991).

Lemma 3. Let f be a $G-$asymptotically nonexpansiveness map in η with asymptotic coefficient ϕ_n such that $\sum_{n=1}^{\infty}(\phi_n - 1) < \infty$. Assume that η has the Property P, then $I - f$ if $G-$demiclosed at 0 (Sangago & et al., 2018).

Lemma 4. Let $\{x_n\}$ be a bounded sequence in reflexive Banacah space X. If for any weakly convergent subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$, both $\left\{x_{n_j}\right\}$ and $\left\{x_{n_{j+1}}\right\}$ converge weakly to the same point in X, then $\{x_n\}$ is weakly convergent (Shahzad&Al-Dubiban, 2006).

The goal of this writing is to furnish a three step iterative procedure to approach of FP of the $G-$ asymptotically nonexpansiveness in *BSWG* and to demonstrate weak and strong convergence results for the proposed maps.

Main Results

Note that $\eta \neq \emptyset \subset X$ is a closed subset of Banach space via $\big(V(G), E(G)\big) = G$ such that $V(G) = \eta$, convex of $E(G)$ and transitive of G. The mappings $f_1, f_2, f_3 : \eta \to \eta$ are $G-$ asymptotically nonexpansiveness maps via $\left\{\phi_n^{(i)}\right\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(\phi_n^{(i)} - 1) < \infty$ for $i = 1,2,3$, resp. Take $\phi_n = max\left\{\phi_n^{(1)}, \phi_n^{(2)}, \phi_n^{(3)}\right\}$ then clearly $\sum_{n=1}^{\infty}(\phi_n - 1) < \infty$. Henceforward we will get the sequence $\{\phi_n\}$ for $\{f_1, f_2, f_3\}$ and $Fix(F) = F(f_1) \cap F(f_2) \cap F(f_3) \neq \emptyset$. For $x_0 \in \eta$, let the sequence $\{x_n\}$ identified by (1.1).

Proposition 1. Let $s^* \in Fix(F)$ be such that $(x_0, s^*), (s^*, x_0) \in E(G)$. Then (x_n, s^*), (s^*, x_n), $(x_n, z_n), (z_n, x_n)$, (s^*, z_n), (z_n, s^*), $(x_n, y_n), (y_n, x_n), (s^*, y_n)$, $(y_n, s^*), (x_n, x_{n+1}) \in E(G)$ for $\forall n \in N$.

Proof. Let $(x_0, s^*) \in E(G)$. We have $(f_1 x_0, s^*) \in E(G)$ in connection with edge preserving of f_1. Since $E(G)$ is convex, we get

$$\begin{aligned}(1 - c_0)(x_0, s^*) + c_0(f_1 x_0, s^*) &= \big((1 - c_0)x_0 + c_0 f_1 x_0, s^*\big) \\ &= (z_0, s^*) \in E(G).\end{aligned}$$

We hold $(f_2 z_0, s^*) \in E(G)$ on the score of edge-preserving f_2. As $(z_0, s^*), (f_2 z_0, s^*) \in E(G)$, $E(G)$ is convex, we have

$$\begin{aligned}(1 - b_0)(z_0, s^*) + b_0(f_2 z_0, s^*) &= \big((1 - b_0)z_0 + b_0 f_2 z_0, s^*\big) \\ &= (y_0, s^*) \in E(G).\end{aligned}$$

By edge-preserving f_2 and f_3, we own $(f_2 x_0, s^*), (f_3 y_0, s^*) \in E(G)$, resp. Due to the convexity of $E(G)$, we obtain

$$\begin{aligned}(1 - a_0)(f_2 x_0, s^*) + a_0(f_3 y_0, s^*) &= \big((1 - a_0)f_2 x_0 + a_0 f_3 y_0, s^*\big) \\ &= (x_1, s^*) \in E(G).\end{aligned}$$

Because of the $\{f_1, f_2, f_3\}$ are edge preserving, we enjoy edge preserving of $\{f_1^2, f_2^2, f_3^2\}$. Thereof, changing (x_1, s^*) in lieu of (x_0, s^*) and $\{f_1^2, f_2^2, f_3^2\}$ in lieu of $\{f_1, f_2, f_3\}$ in above procedure, we get

$$(z_1, s^*), (y_1, s^*) \in E(G) \text{ and } (x_2, s^*) \in E(G).$$

Assume that $(x_u, s^*) \in E(G)$ for $u \in N$. By virtue of the fact that $\{f_1, f_2, f_3\}$ are edge preserving, we acquire edge preserving of $\{f_1^u, f_2^u, f_3^u\}$, and thus $(f_1^u x_u, s^*) \in E(G)$, from the convexity of $E(G)$, we have

$$\begin{aligned}(1 - c_u)(x_u, s^*) + c_u(f_1^u x_u, s^*) &= \big((1 - c_u)x_u + c_u f_1^u x_u, s^*\big) \\ &= (z_u, s^*) \in E(G).\end{aligned}$$

We possess $(f_2^u z_u, s^*) \in E(G)$ in view of edge-preserving f_2^u. Since $(z_u, s^*), (f_2^u z_u, s^*) \in E(G)$, $E(G)$ is convex, we get

$$\begin{aligned}(1-b_u)(z_u, s^*) + b_u(f_2^u z_u, s^*) &= \big((1-b_u)z_u + b_u f_2^u z_u, s^*\big) \\ &= (y_u, s^*) \in E(G).\end{aligned}$$

On account of edge-preserving f_2^u and f_3^u, we own $(f_2^u x_u, s^*), (f_3^u y_u, s^*) \in E(G)$, resp. As $E(G)$ is convex, we obtain

$$\begin{aligned}(1-a_u)(f_2^u x_u, s^*) + a_u(f_3^u y_u, s^*) &= \big((1-a_u)f_2^u x_u + a_u f_3^u y_u, s^*\big) \\ &= (x_{u+1}, s^*) \in E(G).\end{aligned}$$

Resuming the process for $(x_{u+1}, s^*) \in E(G)$, we conclude that

$$(z_{u+1}, s^*), (y_{u+1}, s^*) \in E(G).$$

Hence, by mathematical induction, we obtain that

$$(y_n, s^*), (x_n, s^*), (z_n, s^*) \in E(G) \text{ for } n \in N.$$

Handling an analogue assertion, we could indicate that $(s^*, x_n), (s^*, z_n), (s^*, y_n) \in E(G)$ for $n \in N$, under the supposition that $(s^*, x_0) \in E(G)$. Owing to transitivity of G, we have (x_n, z_n), (z_n, x_n), (x_n, y_n), (y_n, x_n) and $(x_n, x_{n+1}) \in E(G)$ for $n \geq 1$.

Lemma 5. Let X and G be as above. Let $\eta \neq \emptyset$ is a closed convex subset of a real uniformly convex Banach space X. Let $f_1, f_2, f_3 : \eta \to \eta$ are $G-$ asymptotically nonexpansive mappings with $\{\phi_n\} \subset [1, \infty)$ providing $\sum_{n=1}^{\infty}(\phi_n - 1) < \infty$. Assume that $\{x_n\}$ given by (1.1), here $\{a_n\}$, $\{b_n\}, \{c_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$ and $s^* \in Fix(F)$, then

(i) $\lim_{n\to\infty} \|x_n - s^*\|$ exists;
(ii) $\lim_{n\to\infty} \|x_n - f_i x_n\| = 0$ for $i = 1,2,3$.

Proof. (i) Let $s^* \in Fix(F)$. By Proposition 1, (x_n, s^*), (s^*, x_n), $(x_n, z_n), (z_n, x_n)$, (s^*, z_n), (z_n, s^*), $(x_n, y_n), (y_n, x_n), (s^*, y_n)$, $(y_n, s^*)(x_n, x_{n+1})$ are in $E(G)$ for $\forall n \in N$.
From (1.1) and $G-$ asymptotically nonexpansiveness of f_1, we have

$$\begin{aligned}\|z_n - s^*\| &\leq (1-c_n)\|x_n - s^*\| + c_n\|f_1^n x_n - s^*\| \\ &\leq (1-c_n)\|x_n - s^*\| + c_n\phi_n\|x_n - s^*\| \\ &= \{1 + c_n(\phi_n - 1)\|x_n - s^*\|\} \\ &\leq \phi_n\|x_n - s^*\|.\end{aligned} \tag{2.1}$$

It follows from (1.1)&(2.1), $G-$ asymptotically nonexpansiveness of f_2 that we get

$$\begin{aligned}\|y_n - s^*\| &\leq (1-b_n)\|z_n - s^*\| + b_n\|f_2^n z_n - s^*\| \\ &\leq \phi_n\|z_n - s^*\| \leq \phi_n^2\|x_n - s^*\|.\end{aligned} \tag{2.2}$$

Using (1.1)&(2.2), $G-$ asymptotically nonexpansiveness of f_2 and f_3, we have

$$\begin{aligned}\|x_{n+1} - s^*\| &\leq (1-a_n)\|f_2^n x_n - s^*\| + a_n\|f_3^n y_n - s^*\| \\ &\leq \phi_n^3\|x_n - s^*\| \\ &= \{1 + (\phi_n^3 - 1)\}\|x_n - s^*\|.\end{aligned} \tag{2.3}$$

By virtue of $0 \leq \alpha^t - 1 \leq t\alpha^{t-1}(\alpha - 1)$ for all $\alpha \geq 1$, the hypothesis $\sum_{n=1}^{\infty}(\phi_n - 1) < \infty$ implies that $\{\phi_n\}$ is bounded, then $\phi_n \in [1, Q]$ for some Q and $\forall n \geq 1$. Thus, $\phi_n^3 - 1 \leq 3Q^2(\phi_n - 1)$ for $\forall n \geq 1$. Hereat, $\sum_{n=1}^{\infty}(\phi_n^3 - 1) < \infty$. Due to Lemma 1, we find out that $\lim_{n\to\infty} \|x_n - s^*\|$ exists.

(ii) From hypothesis (i), $\lim_{n\to\infty}\|x_n - s^*\|$ exists. Assume that $\lim_{n\to\infty}\|x_n - s^*\| = w$. If $w = 0$, the conclusion is apparent. Suppose $w > 0$. By (2.1)&(2.2), we get

$$limsup_{n\to\infty}\|z_n - s^*\| \leq w, \tag{2.4}$$
$$limsup_{n\to\infty}\|y_n - s^*\| \leq w. \tag{2.5}$$

Hence,

$$\begin{aligned}\|f_1^n x_n - s^*\| &\leq \phi_n\|x_n - s^*\|,\\ \|f_2^n z_n - s^*\| &\leq \phi_n\|z_n - s^*\|,\\ \|f_2^n x_n - s^*\| &\leq \phi_n\|x_n - s^*\|,\\ \|f_3^n y_n - s^*\| &\leq \phi_n\|y_n - s^*\|,\end{aligned}$$

for $\forall n \geq 1$ implies that

$$limsup_{n\to\infty}\|f_1^n x_n - s^*\| \leq w, \tag{2.6}$$
$$limsup_{n\to\infty}\|f_2^n z_n - s^*\| \leq w. \tag{2.7}$$
$$limsup_{n\to\infty}\|f_2^n x_n - s^*\| \leq w, \tag{2.8}$$
$$limsup_{n\to\infty}\|f_3^n y_n - s^*\| \leq w. \tag{2.9}$$

On account of

$$w = \lim_{n\to\infty}\|x_{n+1} - s^*\| = \lim_{n\to\infty}\|(1 - a_n)(f_2^n x_n - s^*) + a_n(f_3^n y_n - s^*)\|$$

by Lemma 2, we have

$$\lim_{n\to\infty}\|f_2^n x_n - f_3^n y_n\| = 0. \tag{2.10}$$

Noting that

$$\begin{aligned}\|x_{n+1} - s^*\| &= \|(1 - a_n)f_2^n x_n + a_n f_3^n y_n - s^*\|\\ &\leq \|f_2^n x_n - s^*\| + a_n\|f_3^n y_n - f_2^n x_n\|\\ &\leq \|f_2^n x_n - s^*\| + [1 - \xi]\|f_3^n y_n - f_2^n x_n\|\end{aligned}$$

which yields that

$$w \leq liminf_{n\to\infty}\|f_2^n x_n - s^*\|. \tag{2.11}$$

By (2.8)&(2.11), we obtain

$$\lim_{n\to\infty}\|f_2^n x_n - s^*\| = w \tag{2.12}$$

Furthermore, from (2.10)&(2.12) and, $G -$ asymptotically nonexpansiveness of f_3

$$\begin{aligned}\|f_2^n x_n - s^*\| &\leq \|f_2^n x_n - f_3^n y_n\| + \|f_3^n y_n - s^*\|\\ &\leq \|f_2^n x_n - f_3^n y_n\| + \phi_n\|y_n - s^*\|\end{aligned}$$

implies that

$$w \leq liminf_{n\to\infty}\|y_n - s^*\|. \tag{2.13}$$

By (2.5)&(2.13), we get

$$\lim_{n\to\infty}\|y_n - s^*\| = w \tag{2.14}$$

Since

$$w = \lim_{n\to\infty}\|y_n - s^*\| = \lim_{n\to\infty}\|(1 - a_n)(z_n - s^*) + a_n(f_2^n z_n - s^*)\|.$$

By (2.4)&(2.7) and Lemma 2, we get

$$\lim_{n\to\infty}\|z_n - f_2^n z_n\| = 0. \qquad (2.15)$$

Noting that

$$\begin{aligned}\|y_n - s^*\| &= \|(1-b_n)z_n + b_n f_2^n z_n - s^*\| \\ &\le \|z_n - s^*\| + b_n\|f_2^n z_n - z_n\| \\ &\le \|z_n - s^*\| + [1-\xi]\|f_2^n z_n - z_n\|\end{aligned}$$

which yields that

$$w \le liminf_{n\to\infty}\|z_n - s^*\|. \qquad (2.16)$$

Using (2.4)&(2.16), we obtain

$$\lim_{n\to\infty}\|z_n - s^*\| = w \qquad (2.17)$$

Because of

$$w = \lim_{n\to\infty}\|z_n - s^*\| = \lim_{n\to\infty}\|(1-c_n)(x_n - s^*) + c_n(f_1^n x_n - s^*)\|$$

from Lemma 2, we get

$$\lim_{n\to\infty}\|x_n - f_1^n x_n\| = 0. \qquad (2.18)$$

On the other hand, by (2.18)

$$\begin{aligned}\|z_n - x_n\| &= \|(1-c_n)x_n + c_n f_1^n x_n - x_n\| \qquad (2.19)\\ &\le c_n\|f_1^n x_n - x_n\| \\ &\le [1-\xi]\|f_1^n x_n - x_n\| \\ &\to 0 \; as \; n \to \infty,\end{aligned}$$

and, from (2.15)&(2.19), we have

$$\begin{aligned}\|y_n - x_n\| &= \|(1-b_n)z_n + b_n f_2^n z_n - x_n\| \qquad (2.20)\\ &\le \|z_n - x_n\| + b_n\|f_2^n z_n - z_n\| \\ &\le \|z_n - x_n\| + [1-\xi]\|f_2^n z_n - z_n\| \\ &\to 0 \; as \; n \to \infty,\end{aligned}$$

Using (2.15)&(2.19), we obtain

$$\begin{aligned}\|x_n - f_2^n x_n\| &\le \|z_n - x_n\| + \|f_2^n z_n - z_n\| + \|f_2^n z_n - f_2^n x_n\| \\ &\le \|z_n - x_n\| + \|f_2^n z_n - z_n\| + \phi_n\|z_n - x_n\|\end{aligned}$$

so that

$$\|x_n - f_2^n x_n\| \to 0 \; as \; n \to \infty. \qquad (2.21)$$

Also

$$\begin{aligned}\|x_n - f_3^n x_n\| &\le \|x_n - f_2^n x_n\| + \|f_2^n x_n - f_3^n y_n\| + \|f_3^n y_n - f_3^n x_n\| \\ &\le \|x_n - f_2^n x_n\| + \|f_2^n x_n - f_3^n y_n\| + \phi_n\|y_n - x_n\|\end{aligned}$$

implies by (2.10), (2.20)&(2.21) that

$$\|x_n - f_3^n x_n\| \to 0 \; as \; n \to \infty. \qquad (2.22)$$

It follows from (2.10)&(2.21) that we get

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-a_n)f_2^n x_n + a_n f_3^n y_n - x_n\| \qquad (2.23)\\ &\le \|f_2^n x_n - x_n\| + a_n\|f_3^n y_n - f_2^n x_n\| \\ &\le \|f_2^n x_n - x_n\| + [1-\xi]\|f_3^n y_n - f_2^n x_n\|\end{aligned}$$

$\to 0 \; as \; n \to \infty.$

Then, for $l = 1,2,3$

$$\begin{aligned}\|x_n - f_l x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - f_l^{n+1}x_{n+1}\| + \|f_l^{n+1}x_{n+1} - f_l^{n+1}x_n\| \\ &\quad + \|f_l^{n+1}x_n - f_l x_n\| \\ &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - f_l^{n+1}x_{n+1}\| + \phi_{n+1}\|x_{n+1} - x_n\| \\ &\quad + \phi_1\|f_l^n x_n - x_n\|\end{aligned}$$

by (2.18), (2.21), (2.22)&(2.23) gives that for $l = 1,2,3$

$$\lim_{n\to\infty}\|x_n - f_l x_n\| = 0. \tag{2.24}$$

Theorem 1. Let X, η, G and f_l for $l = 1,2,3$ be as above. Suppose that η has the Property P, f_l for $l = 1,2,3$ satisfy the *Condition (C)*, $\{x_n\}$ defined by (1.1), where $\{a_n\}$, $\{b_n\},\{c_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \to s^* \in Fix(F)$.

Proof. Let $s^* \in Fix(F)$ such that (x_n, s^*), (s^*, x_n), $(x_n, z_n), (z_n, x_n)$, (s^*, z_n), (z_n, s^*), $(x_n, y_n), (y_n, x_n), (s^*, y_n)$, $(y_n, s^*)(x_n, x_{n+1})$ are in $E(G)$ for $\forall n \in N$. From Lemma 5 (i), $\lim_{n\to\infty}\|x_n - s^*\|$ exists, and thus $\lim_{n\to\infty} d(x_n, Fix(F))$ exists for all $s^* \in Fix(F)$. Again, by Lemma 5 (ii) $\lim_{n\to\infty}\|x_n - f_l x_n\| = 0$ for $l = 1,2,3$. Using the Condition (C), there is a nondecreasing function $g: [0,\infty) \to [0,\infty)$ with $g(t) > 0$ for $\forall t \in (0,\infty)$, $g(0) = 0$ such that $max\{\|x - f_1 x\|, \|x - f_2 x\|, \|x - f_3 x\|\} \geq g\left(d\big(x, Fix(F)\big)\right)$, that is to say

$$\lim_{n\to\infty} g\left(d\big(x, Fix(F)\big)\right) = 0. \tag{2.25}$$

Hereby, we can receive a subsequence $\{x_{n_s}\}$ and $\{x_n\}$ and sequence $\{\alpha_s\} \subset Fix(F)$ such that $\|x_{n_s} - \alpha_s\| < 2^{-s}$ for $\forall s \geq 1$. Because of the fact that strong convergence stand for weak convergence and by Remark 1, we hold $(x_{n_s}, \alpha_s) \in E(G)$. By the proof of [23], we have $\|x_{n_{s+1}} - \alpha_s\| < \|x_{n_s} - \alpha_s\| < 2^{-s}$, and so $\|\alpha_{s+1} - \alpha_s\| < {}^{2}/_{2^s}$. We infer that $\{\alpha_s\}$ is a Cauchy sequence in $Fix(F)$, thus it converges. Let $\alpha_s \to \alpha$. As $Fix(F)$ is closed, and thus $\alpha \in Fix(F)$ and then $x_{n_s} \to \alpha$. From Lemma 5 (i), $x_n \to s^* \in Fix(F)$.

In an analog way to Theorem 1, when $c_n \equiv 0$ in (1.1), we show convergence result of a modified $S-$iterative procedure as shown below.

Theorem 2. Let X, η, G and f_2, f_3 be as above. Suppose that η has the Property P, f_2, f_3 satisfy the *Condition (B)*, $\{x_n\}$ defined by (1.1), where $\{a_n\}$, $\{b_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \to s^* \in F(f_2) \cap F(f_3)$.

In line with the Theorem 1, if $f_2 = I$ and $b_n = c_n \equiv 0$ in (1.1), then obtain Mann-type convergence results for $G-$asymptotically nonexpansive mappings with *BSWG* as noted below.

Theorem 3. Let X, η, G and f_3 be as above. Suppose that η has the Property P, f_3 satisfy the *Condition (A)*, $\{x_n\}$ defined by (1.1), where $\{a_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \to s^* \in F(f_3)$.

In the next theorem, we express the strong convergence of $\{x_n\}$ defined by (1.1) under semi-compact.

Theorem 4. Let X, η, G and f_l for $l = 1,2,3$ be as above. Suppose that η has the Property P, one of f_l for $l = 1,2,3$ is *semi-compact*, $\{x_n\}$ defined by (1.1), where $\{a_n\}, \{b_n\}, \{c_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \to s^* \in Fix(F)$.

Proof. Let $s^* \in Fix(F)$ such that (x_n, s^*), (s^*, x_n), $(x_n, z_n), (z_n, x_n)$, (s^*, z_n), (z_n, s^*), $(x_n, y_n), (y_n, x_n), (s^*, y_n)$, $(y_n, s^*)(x_n, x_{n+1})$ are in $E(G)$ for $\forall n \in N$. By virtue of the fact that one of f_l for $l = 1,2,3$ is semi-compact, $\{x_n\}$ is bounded and by (2.24), and then

$$there\ exists\ subsequence\ \left\{x_{n_p}\right\}\ of\ \{x_n\}\ such\ that\ x_{n_p} \to s^*. \tag{2.26}$$

Owing to the fact that strong convergence state weak convergence and by Remark 1, we enjoy $\left(x_{n_p}, s^*\right) \in E(G)$. We get that $s^* \in Fix(F)$. Hereby, $\lim_{n\to\infty} \|x_n - s^*\|$ exists from Lemma 5 (i). On the score of (2.26), then $\{x_n\} \to s^* \in Fix(F)$.

For $c_n \equiv 0$ in (1.1), then Theorem 4, reduces to the following convergence result of a modified S −iterative procedure for G −asymptotically nonexpansiveness in *BSWG*.

Theorem 5. Let X, η, G and f_l for $l = 2,3$ be as above. Suppose that η has the Property P, one of f_l for $l = 2,3$ is *semi-compact*, $\{x_n\}$ defined by (1.1), where $\{a_n\}$, $\{b_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \to s^* \in F(f_2) \cap F(f_3)$.

Similarly, taking $f_2 = I$ and $b_n = c_n \equiv 0$ in Theorem 4, then we attain Mann-type convergence results for G −asymptotically nonexpansive mappings with *BSWG* as follows.

Theorem 6. Let X, η, G and f_3 be as above. Suppose that η has the Property P, f_3 is semi-compact, $\{x_n\}$ defined by (1.1), where $\{a_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \to s^* \in F(f_3)$.

In the subsequent result, we testify the weak convergence of iteration (1.1) for G −asymptotically nonexpansiveness with *BSWG* without assuming the Opial's condition of X.

Theorem 7. Let X, η, G and f_l for $l = 1,2,3$ be as above. Suppose that η has the Property P, $\{x_n\}$ defined by (1.1), where $\{a_n\}$, $\{b_n\}, \{c_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \rightharpoonup u_* \in Fix(F)$.

Proof. Let $s^* \in Fix(F)$ such that (x_n, s^*), (s^*, x_n), $(x_n, z_n), (z_n, x_n)$, (s^*, z_n), (z_n, s^*), $(x_n, y_n), (y_n, x_n), (s^*, y_n)$, $(y_n, s^*)(x_n, x_{n+1})$ are in $E(G)$ for $\forall n \in N$. $\{x_n\}$ is bounded from Lemma 5 (i). Because of that $\eta \neq \emptyset$ is a closed convex subset of a real uniformly convex Banach space X, it is weakly compact, and thus there exists subsequence $\left\{x_{n_p}\right\}$ of $\{x_n\}$ such that $x_{n_p} \to u_*$. We know that for $l = 1,2,3$ $\lim_{p\to\infty} \left\|x_{n_p} - f_l x_{n_p}\right\| = 0$ by (2.24). Using Lemma 3, $I - f_l$ for $l = 1,2,3$ is $G-$ demiclosed at 0 so that $u_* \in Fix(F)$.

Now, we show that $x_n \rightharpoonup u_*$. Let $\left\{x_{n_p}\right\}$ be a subsequence of $\{x_n\}$ that converges weakly to $w_* \in K$. On the lines similar to above, we also have $w_* \in Fix(F)$. Next, for each $p \geq 1$, we get

$$\begin{aligned} x_{n_p+1} &= \left(1 - a_{n_p}\right) f_2^{n_p} x_{n_p} + a_{n_p} f_3^{n_p} y_{n_p} - x_{n_p} + x_{n_p} \\ &= x_{n_p} + \left(f_2^{n_p} x_{n_p} - x_{n_p}\right) + a_{n_p}\left(f_3^{n_p} y_{n_p} - f_2^{n_p} x_{n_p}\right). \end{aligned} \tag{2.27}$$

It follows from (2.10) and (2.21) that

$$\lim_{p\to\infty} \left\|f_3^{n_p} y_{n_p} - f_2^{n_p} x_{n_p}\right\| = 0\ and\ \lim_{p\to\infty} \left\|f_2^{n_p} x_{n_p} - x_{n_p}\right\| = 0. \tag{2.28}$$

Due to $a_{n_p} \in [\xi, 1-\xi]$ for some $\xi \in (0,1)$, we obtain

$$\lim_{p\to\infty} a_{n_p} \left\| f_3^{n_p} y_{n_p} - f_2^{n_p} x_{n_p} \right\| = 0\,; \tag{2.29}$$

vincelet, using (2.28) and (2.29), we deduce that $x_{n_p+1} \rightharpoonup w_*$. By Lemma 4, $\{x_n\} \rightharpoonup w_*$ so that $w_* = u_*$.

Putting $c_n \equiv 0$ in (1.1), then Theorem 7 reduces to the following convergence result of a modified S −iterative procedure for G −asymptotically nonexpansiveness in *BSWG*.

Theorem 8. Let X, η, G and f_l for $l = 2,3$ be as above. Suppose that η has the Property P, $\{x_n\}$ defined by (1.1), where $\{a_n\}$, $\{b_n\}, \{c_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \rightharpoonup u_* \in F(f_2) \cap F(f_3)$.

In a similar manner, when $f_2 = I$ and $b_n = c_n \equiv 0$ in Theorem 7, then we get Mann-type convergence results for G −asymptotically nonexpansiveness in *BSWG* as follows.

Theorem 9. Let X, η, G and f_3 be as above. Suppose that η has the Property P, $\{x_n\}$ defined by (1.1), where $\{a_n\}, \{b_n\}, \{c_n\} \subset [\xi, 1-\xi]$ for some $\xi \in (0,1)$ and $(x_0, s^*), (s^*, x_0) \in E(G)$ for $x_0 \in \eta$, then $\{x_n\} \rightharpoonup u_* \in F(f_2) \cap F(f_3)$.

Remark 2. If $\phi_n \equiv 0$ for $\forall n \geq 1$, then the class of G −asymptotically nonexpansiveness coincide with G −nonexpansiveness. Because of this reason, Theorem 1-7 widen and enhance the results of Theorem 2-3 in Suparatulatorn& et al. and Theorem 3.12-3.15 in Thakur& et al ((Suparatulatorn& et al., 2018), (Thakur& et al., 2014)).

Now, we consider the numerical instance which is inspired by Example 3.1 in Razani&Moradi (Razani&Moradi, 2015).

Example 1. Let X is the real line with the usual norm $|.|$, $\eta = [0,2]$ and $G = (V(G), E(G))$ such that $(V(G) = \eta$ and $(x,y) \in E(G)$ iff $0.30 \leq x, y \leq 1.20$ or $x = y$. Define $f_l: \eta \to \eta$ for $l = 1,2,3$ as

$$f_1 x = \begin{cases} \dfrac{3\sin x}{10+10x}, & x = 0.5 \\ 0, & x \neq 0.5 \end{cases},$$

$$f_2 x = \begin{cases} \dfrac{2\sin x}{5+5x}, & x = 0.5 \\ 0, & x \neq 0.5 \end{cases}$$

$$f_3 x = \begin{cases} \dfrac{\sin x}{10+10x}, & x = 0.5 \\ 0, & x \neq 0.5 \end{cases}$$

for any $x \in \eta$. It is easy to see that f_l for $l = 1,2,3$ are G −asymptotically nonexpansive mappings. Set

$$a_n = b_n = c_n = 1/2\,, for\ n \geq 1.$$

Let $\{x_n\}$ be the sequence defined by (1.1). Thereof,

$$\begin{aligned} x_{n+1} &= 1/2\, f_2^n x_n + 1/2\, f_3^n y_n, \\ y_n &= 1/2\, z_n + 1/2\, f_2^n z_n, \\ z_n &= 1/2\, x_n + 1/2\, f_1^n x_n, n \geq 1, \end{aligned}$$

and $Fix(F) = \{0\}$. As $x_1 = 0.5$, we know that $x_2 = 7.2003 \times 10^{-2}$, $x_3 = 5.3372 \times 10^{-3}$, $x_4 = 1.7011 \times 10^{-4}$, $x_5 = 2.179 \times 10^{-6}$, $x_6 = 1.1159 \times 10^{-8}$ and $x_7 = 2.2855 \times 10^{-11}$ (see Figure 1). This example reveals that the algorithm is efficient to approach common fixed points of G −asymptotically nonexpansiveness on *BSWG*.

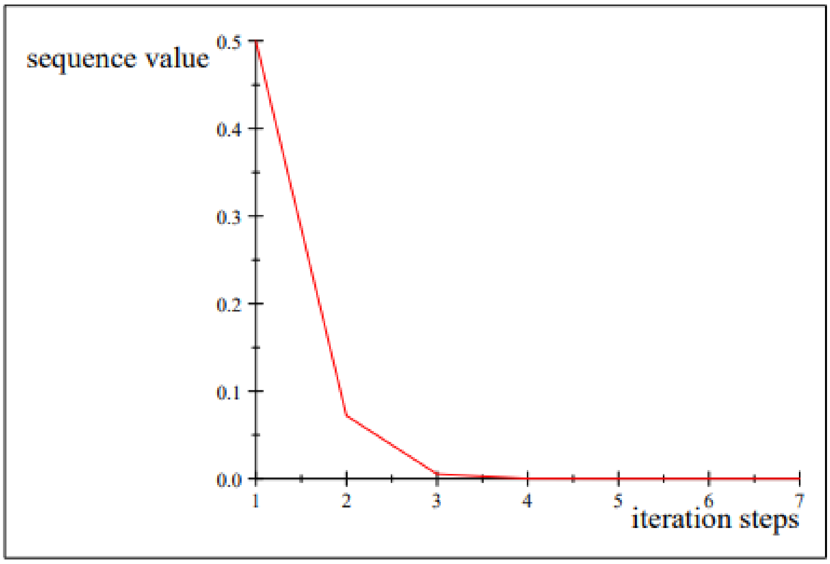

Figure 1. The values in the first seven steps of $\{x_n\}$ in Example 1.

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